

A Comparative Study of Conventional Nonsmooth Time-Invariant and Smooth Time-Varying Robust Compensators

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Abstract— Variable structure-based robust compensators are general approaches to deal with uncertain nonlinear dynamic systems. The bang-bang (discontinuous-type) compensator and the saturation (continuous-type) compensator are the two common robust compensators used in variable structure controllers. This paper presents a comparative study of these two nonsmooth time-invariant robust compensators with a smooth time-varying compensator and reveals the superiority of the smooth time-varying compensator over the nonsmooth robust compensators.

Index Terms— Bang-bang control, compensation, robustness, smooth compensator, stability, uniform ultimate boundedness, variable structure systems.

I. INTRODUCTION

IN THE COURSE of designing controllers for dynamic systems, one has to often deal with uncertainties. Variable structure-based robust compensators are generally used in such situations for nonlinear systems [12], [13], [8]. The dominant role in variable structure system (VSS) theory is played by sliding modes and the core idea of designing control algorithms consists of enforcing the system trajectories onto the sliding surface [14]. Implementation of sliding mode control using robust compensators implies switching. The conventional robust compensators employ control action which switches around the sliding surface in either a continuous or a discontinuous manner. The compensator with discontinuous switching is called the discontinuous-type or the bang-bang compensator and the one with continuous switching is called the continuous-type or the saturation compensator. Theoretically, the bang-bang compensator guarantees the asymptotic stability of the closed-loop system, [5], [6], for example, but from a practical standpoint it often excites the high-frequency modes and the unmodeled dynamics of the system, resulting in chattering. This can be attributed to the discontinuous nature of the compensator.

The continuous compensator can improve the system performance by introducing a boundary layer that reduces chattering. However, it cannot achieve asymptotic stability—it can only guarantee the convergence of the system trajectory to a bounded neighborhood of the origin. This is inadequate for some applications, such as in high-precision manufacturing. It should also be noted that the saturation compensator is continuous but not smooth. The nonsmooth nature of the

compensator is undesirable in applications such as in the control of space structures [7].

It becomes clear that both the bang-bang and the saturation compensators have their drawbacks. While using these compensators, there is a certain tradeoff between accuracy and robustness to unmodeled high-frequency dynamics. This has motivated the development of a smooth time-varying robust compensator [1], [2] which guarantees both asymptotic stability of the closed-loop system and smooth control action, in order to improve the control accuracy while avoiding chattering. The smooth robust compensator has been successfully applied to a number of problems involving uncertainty compensation, such as trajectory tracking of robots with joint friction [3], control of robots with flexible joints [10], and control of a single-degree-of-freedom (SDOF) servo system with stick-slip friction [11]. However, the comparison of the smooth compensator with the bang-bang and the saturation compensator has not been carried out. In this paper, we demonstrate the advantages of the smooth time-varying robust compensator over the conventional compensators with the help of a numerical simulation. Also, the asymptotic stability of a closed-loop uncertain dynamical system with the smooth time-varying robust compensator is theoretically proved using Lyapunov's direct method along with the concept of ultimate boundedness.

The rest of this paper is organized as follows. In Section II, we review the conventional robust compensators. In Section III, we present the smooth time-varying robust compensator [1], [2] and show that it can guarantee asymptotic stability of an uncertain dynamic system. In this section we also discuss the advantages of the smooth compensator. The efficacy of the compensator is demonstrated in Section IV through numerical simulations of a simple SDOF system. Concluding remarks are provided in Section V.

II. REVIEW OF CONVENTIONAL ROBUST COMPENSATORS

Consider the tracking control of a SDOF uncertain mechanical system described by the dynamics

$$M\ddot{x}(t) = u \quad (1)$$

where x is the generalized coordinate of the system, M is the generalized inertia, and u is the generalized force and the control input. Let us assume the generalized inertia M to be unknown but upper bounded. Let \hat{M} denote the estimated value of M and assume that there exists a positive number β , which satisfies

$$\beta > |\hat{M}| \quad (2)$$

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where \tilde{M} is defined as $\tilde{M} \triangleq (\hat{M} - M)$ and denotes the uncertainty in the parameter M . The controller for the uncertain system described by (1) employing a bang-bang compensator can be proposed as

$$u = \hat{M}\ddot{x}_d - \hat{M}\lambda\dot{e} - k_d s - \rho \operatorname{sign}(s) \quad (3)$$

where x_d is the desired trajectory, e is the tracking error defined as $e \triangleq (x - x_d)$, λ , and k_d are positive numbers, and ρ , s , and $\operatorname{sign}(s)$ are defined as

$$\rho \triangleq \beta|\ddot{x}_d - \lambda\dot{e}| \quad (4)$$

$$s \triangleq (\dot{e} + \lambda e) \quad (5)$$

$$\operatorname{sign}(s) \triangleq \begin{cases} 1, & \text{if } s > 0 \\ -1, & \text{if } s < 0. \end{cases} \quad (6)$$

Also, $s = 0$ defines the sliding surface on which system trajectories are globally asymptotically stable. In (3), $\hat{M}\ddot{x}_d$ represents a feedforward input, $-\hat{M}\lambda\dot{e}$ and $-k_d s$ together constitute a proportional plus derivative (PD) controller and $-\rho \operatorname{sign}(s)$ is the bang-bang compensator designed to compensate for the uncertainty in the system inertia M and to ensure zero tracking error of the system.

The controller in (3) guarantees the asymptotic stability of the dynamic system in (1). This can be easily proved by using the Lyapunov function $V = Ms^2$. The bang-bang compensator in (3) however leads to chattering and therefore it is a common practice to use the saturation compensator to improve the system performance.

The saturation compensator introduces a boundary layer in the neighborhood of the sliding surface and reduces chattering at the expense of asymptotic stability. For the system in (1), the saturation compensator-based controller can be proposed as

$$u = \hat{M}\ddot{x}_d - \hat{M}\lambda\dot{e} - k_d s - \rho \operatorname{sat}\left(\frac{s}{\phi}\right) \quad (7)$$

where

$$\operatorname{sat}\left(\frac{s}{\phi}\right) \triangleq \begin{cases} 1, & \text{if } s > \phi \\ -1, & \text{if } s < -\phi \\ s/\phi, & \text{if } |s| < \phi \end{cases} \quad (8)$$

and where ρ is defined by (4). By constructing a Lyapunov function $V = Ms^2$, it is possible to show that all system trajectories are ultimately bounded [4] and can converge to a neighborhood of the sliding surface.

III. THE SMOOTH TIME-VARYING ROBUST COMPENSATOR

Instead of using the bang-bang compensator or the saturation compensator, a smooth time-varying compensator [1], [2] can be employed for controlling the uncertain system in (1) as follows:

$$u = \hat{M}\ddot{x}_d - \hat{M}\lambda\dot{e} - k_d s - \rho \tanh[(a + bt)s] \quad (9)$$

where ρ has been defined in (4), a and b are positive numbers and the hyperbolic tangent function is defined as

$$\tanh(*) = \frac{e^{*} - e^{-*}}{e^{*} + e^{-*}}.$$

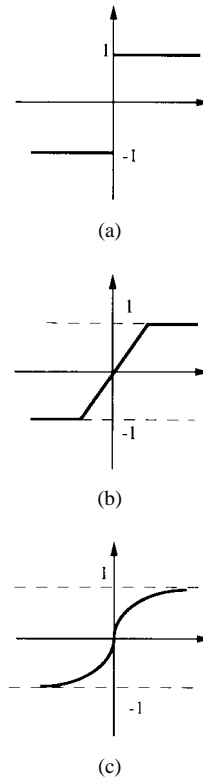


Fig. 1. Comparison of the control action of different robust compensators, namely, (a) the bang-bang compensator, (b) the saturation compensator, and (c) the smooth compensator.

Remark 1: The robust compensator in (9) is continuously differentiable with respect to both time t , and the control variable s . Therefore, it generates a smooth control action. A comparison of the control action produced by the proposed compensator with those produced by the conventional compensators is shown in Fig. 1.

The stability of the system in (1) under the control given by (9) is formally stated next with the help of the following theorem.

Theorem: The system in (1) under the control given by (9) is globally uniformly asymptotically stable, and therefore, $e, \dot{e} \rightarrow 0$, as $t \rightarrow \infty$.

Proof: The fact that $e, \dot{e} \rightarrow 0$, as $t \rightarrow \infty$ can be proved if we can show that $s \rightarrow 0$ as $t \rightarrow \infty$ [9]. To show that $s \rightarrow 0$ as $t \rightarrow \infty$, we will first derive the closed-loop system in terms of s .

We substitute (9) into (1) to obtain the closed-loop system

$$M\ddot{x}(t) = \hat{M}\ddot{x}_d - \hat{M}\lambda\dot{e} - k_d s - \rho \tanh[(a + bt)s]. \quad (10)$$

Using the relation $\tilde{M} \triangleq (\hat{M} - M)$ and the relation in (5), (10) can be modified to the form

$$M\dot{s} + k_d s = -\rho \tanh[(a + bt)s] + \tilde{M}(\ddot{x}_d - \lambda\dot{e}). \quad (11)$$

In order to demonstrate the asymptotic stability of the closed-loop system defined by (11), we now propose the

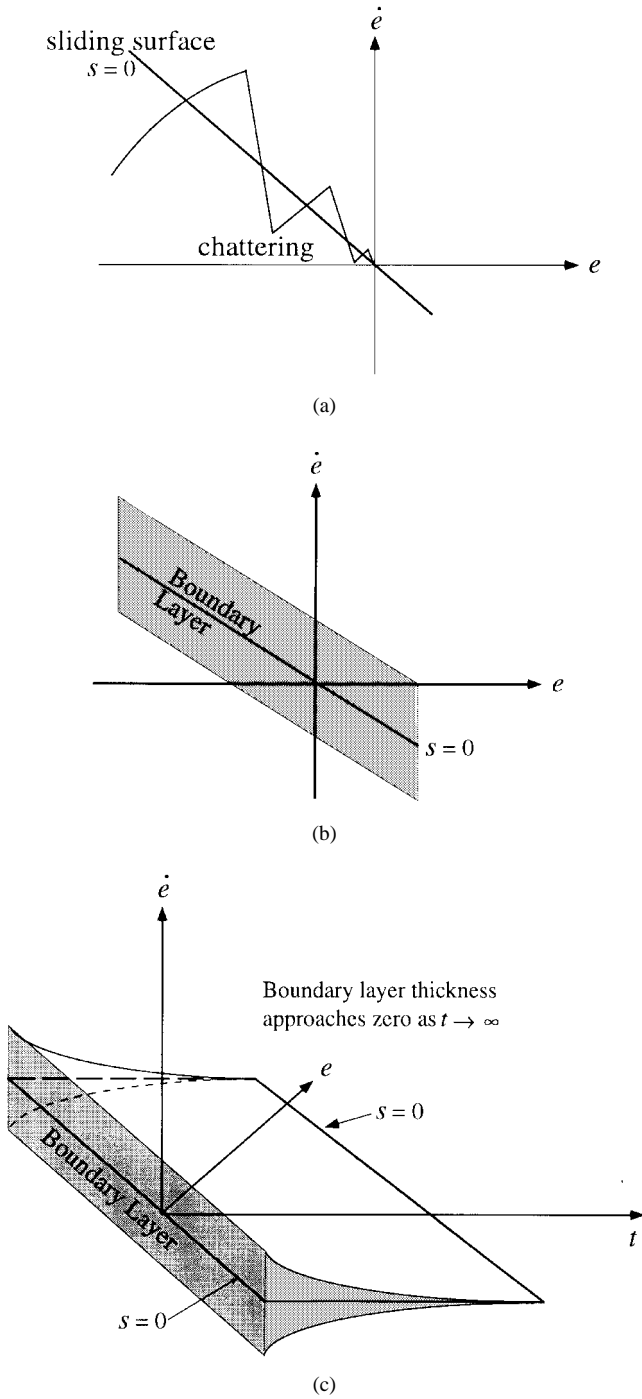


Fig. 2. The sliding surface and its associated boundary layer for the cases (a) the bang-bang compensator, (b) the saturation compensator, and (c) the smooth compensator.

following Lyapunov function candidate:

$$V = \frac{1}{2} M s^2. \quad (12)$$

The function V satisfies

- 1) $V \geq 0$;
- 2) $V \rightarrow \infty$ uniformly for $t > 0$ as $|s| \rightarrow \infty$;
- 3) V has continuous partial derivative with respect to s .

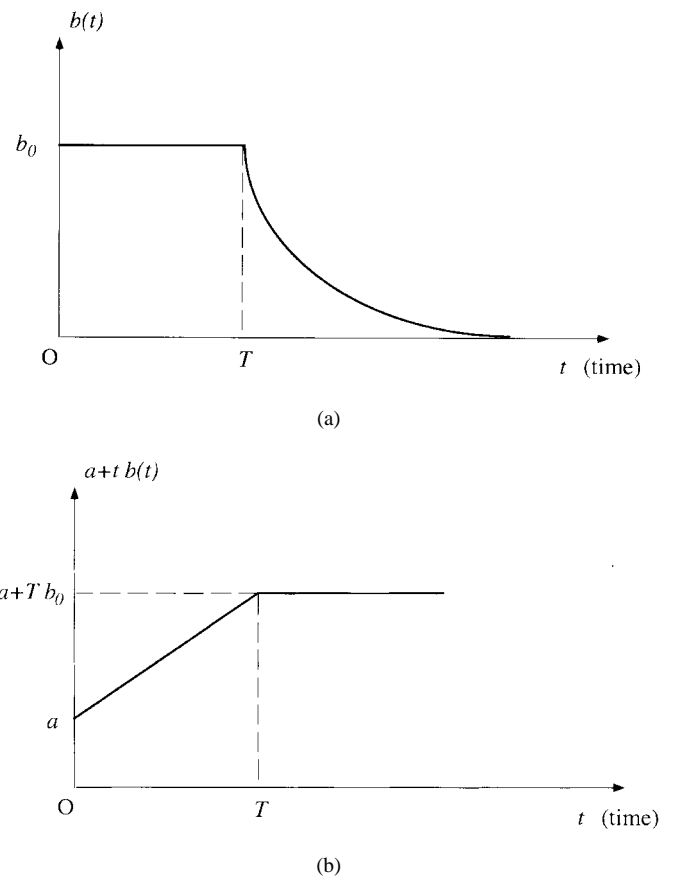


Fig. 3. (a) A possible choice of $b(t)$ for practical implementation. (b) Plot of $(a + bt)$ for the choice of $b(t)$ shown in Fig. 3(a).

The derivative of V along the system trajectory given by (11) is computed as

$$\begin{aligned} \dot{V} &= sM\dot{s} \\ &= s[-k_d s - \rho \tanh\{(a + bt)s\} + \tilde{M}(\ddot{x}_d - \lambda\dot{e})] \\ &\leq -k_d s^2 + |s| |\tilde{M}| |\ddot{x}_d - \lambda\dot{e}| - \rho |s| \tanh\{(a + bt)|s|\} \\ &= -k_d s^2 - \rho |s| \left[\tanh\{(a + bt)|s|\} - \frac{1}{\rho} |\tilde{M}| |\ddot{x}_d - \lambda\dot{e}| \right] \\ &= -k_d s^2 - \rho |s| [\tanh\{(a + bt)|s|\} - \alpha] \end{aligned} \quad (13)$$

where the inequality in (2) and (4) were used. The quantity α is defined as

$$\alpha \triangleq \frac{1}{\rho} |\tilde{M}| |\ddot{x}_d - \lambda\dot{e}|. \quad (14)$$

From (14) and from the definition of ρ in (4), we know that

$$0 \leq \alpha < 1. \quad (15)$$

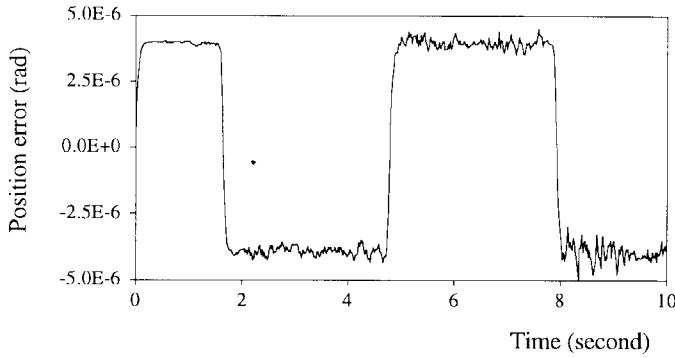
It is clear from (13) that $\dot{V} < 0$ as long as

$$\tanh\{(a + bt)|s|\} > \alpha. \quad (16)$$

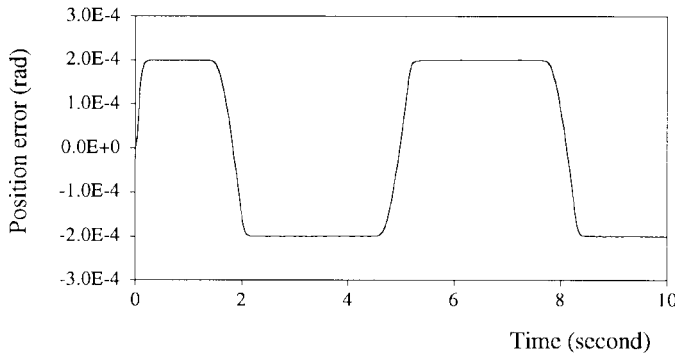
The inequality in (16) implies that $\dot{V} < 0$ if

$$|s| > \gamma(t), \quad \gamma(t) \triangleq \frac{1}{2(a + bt)} \ln \left(\frac{1 + \alpha}{1 - \alpha} \right). \quad (17)$$

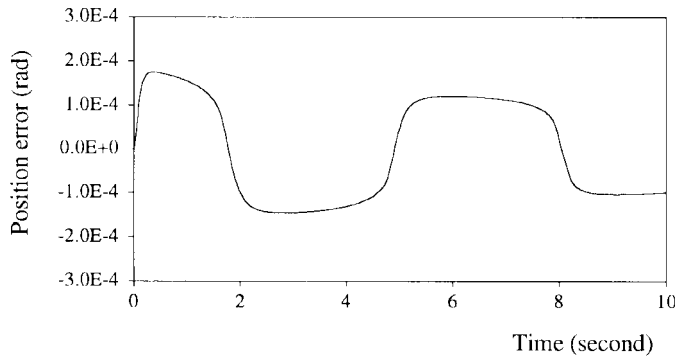
Define the closed set $\Sigma = \{s \in R: |s| \leq \gamma(t)\}$ and let $\Sigma_\epsilon \supset \Sigma$ denote the set of all points whose distance from Σ is



(a)

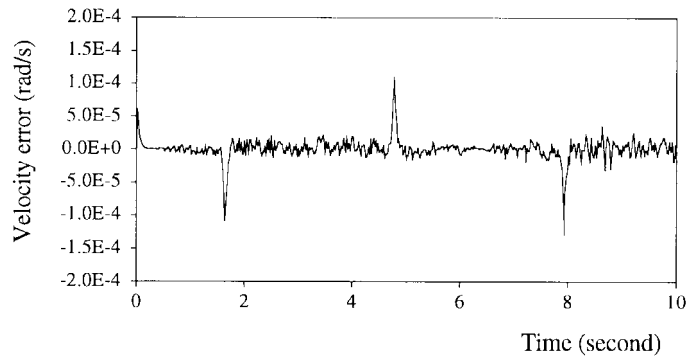


(b)

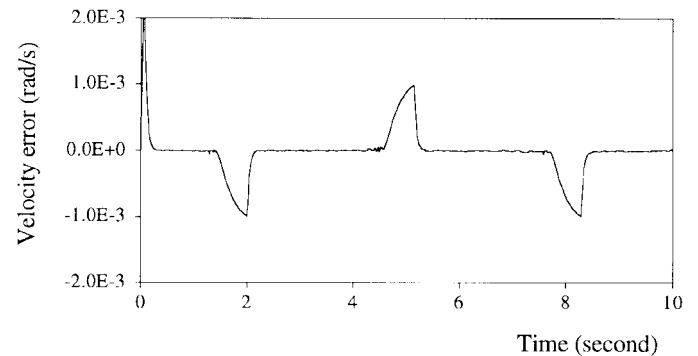


(c)

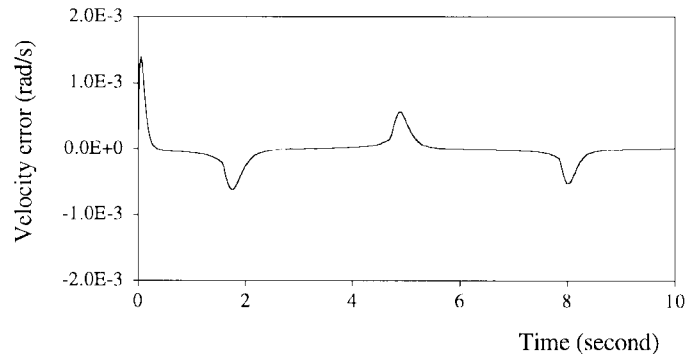
Fig. 4. Comparison of position errors with different robust compensators. (a) The bang-bang compensator. (b) The saturation compensator. (c) The smooth compensator.



(a)



(b)



(c)

Fig. 5. Comparison of velocity errors with different robust compensators. (a) The bang-bang compensator. (b) The saturation compensator. (c) The smooth compensator.

less than ϵ for a small positive number ϵ . Let Σ^C denote the complement of Σ and Σ_ϵ^C denote the complement of Σ_ϵ . Now it can be shown very simply from (12) and (13) that V and its derivative additionally satisfy:

- 1) $V(s_1) < V(s_2)$ for all $s_1 \in \Sigma$ and all $s_2 \in \Sigma_\epsilon^C$;
- 2) $\dot{V}(s, t) \leq -\delta$ for all $t \geq 0$ and all $s \in \Sigma^C$, where $\delta > 0$.

Therefore the trajectories of the system in (11) are globally uniformly ultimately bounded [4]. This implies that within a finite time, the trajectories of s will be confined within the set Σ_ϵ .

From the definition of $\gamma(t)$ in (17), it is clear that $\gamma(t) \rightarrow 0$ as $t \rightarrow \infty$. Therefore, in the limit as time $t \rightarrow \infty$, the closed set Σ contains only the origin $s = 0$ and the derivative of the

Lyapunov function satisfies $\dot{V} < 0$ at all points other than the origin and $\dot{V} = 0$ at the origin.

Since the trajectories of (11) are globally uniformly ultimately bounded and since $\dot{V} = 0$ only at the origin when $t \rightarrow \infty$, the trajectories of the system in (11) converge onto the sliding surface and are globally asymptotically stable. This concludes the proof.

Remark 2: The time-varying nature of the proposed compensator guarantees the asymptotic stability of the closed-loop system. This can be interpreted from the viewpoint of the sliding surface and its associated boundary layer. The bang-bang compensator has no boundary layer, as shown in Fig. 2(a), and therefore it can ensure asymptotic stability. The saturation compensator, shown in Fig. 2(b), has a boundary

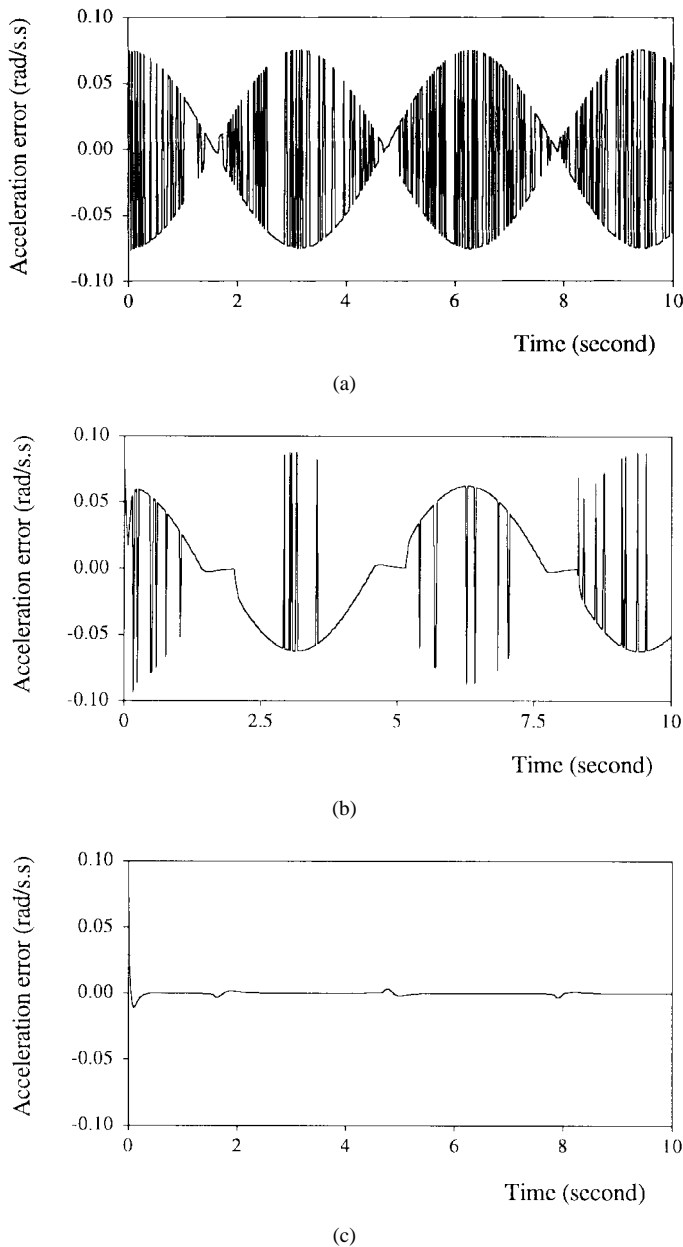


Fig. 6. Comparison of acceleration errors with different robust compensators. (a) The bang-bang compensator. (b) The saturation compensator. (c) The smooth compensator.

layer and therefore performance is enhanced at the expense of asymptotic stability. The proposed compensator, shown in Fig. 2(c), has a time-varying boundary layer whose thickness $\gamma(t)$ depends on a and b and approaches zero as time $t \rightarrow \infty$. Therefore asymptotic stability can be ensured without sacrificing system performance.

Remark 3: It is worthwhile to point out that the time-varying hyperbolic tangent compensator $\tanh\{(a + bt)|s|\}$ does not behave like a bang-bang compensator in the limit as $t \rightarrow \infty$. This can be explained as follows. From the proof of the theorem we know that the trajectories of s are globally uniformly ultimately bounded and therefore as time t becomes large, $|s|$ approaches a ball of radius $\gamma(t)$ centered at the origin. Since $\gamma(t)$ decays as the reciprocal of $(a + bt)$, as seen from

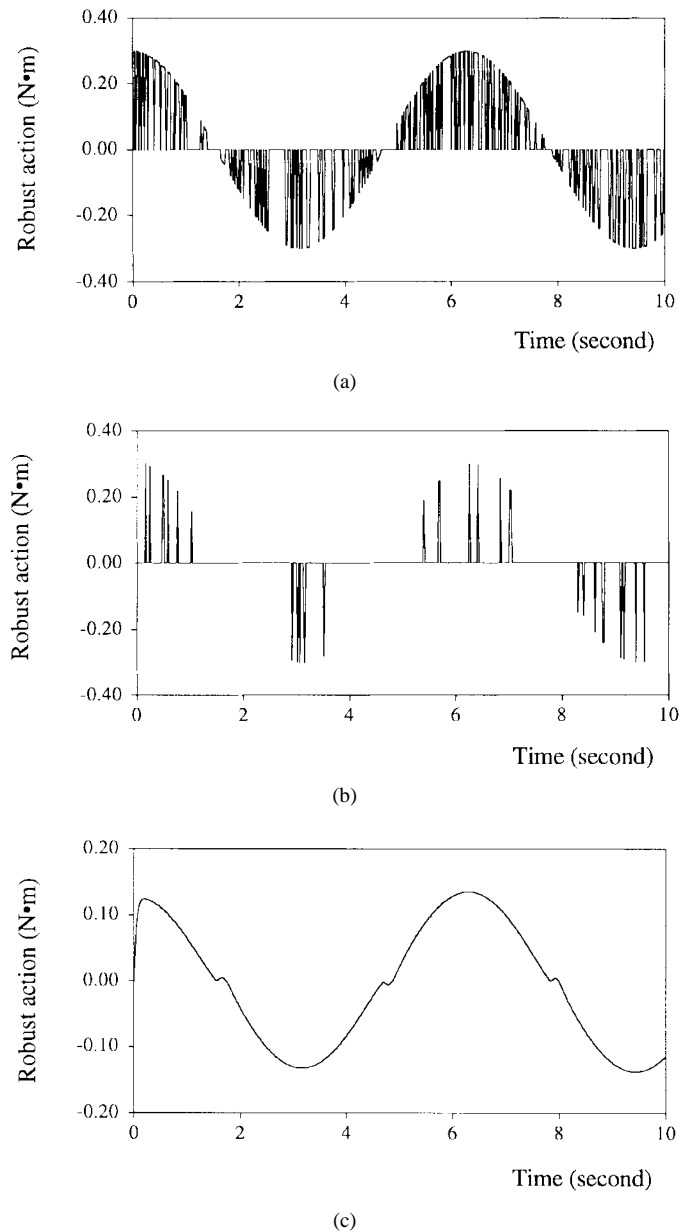


Fig. 7. Comparison of robust control action with different robust compensators. (a) The bang-bang compensator. (b) The saturation compensator. (c) The smooth compensator.

(17), the argument of the hyperbolic tangent function remains small. Thus no bang-bang action is produced in the limit.

Remark 4: In a practical situation, it will be impossible to achieve zero error due to actuator and sensor limitations. To control the error within a desired accuracy, one might consider the modification of the compensator to the following form:

$$\tanh\{(a + b(t)t)s\}$$

where one choice of $b(t)$ can be

$$b(t) = \begin{cases} b_0, & \text{if } 0 \leq t \leq T \\ b_0(T/t), & \text{if } t > T. \end{cases}$$

This particular choice of $b(t)$ is plotted in Fig. 3(a). In this choice of $b(t)$, a , b_0 , and T are positive numbers that are to

be chosen according to the desired accuracy and the rate of convergence. The term $(a + b(t)t)$, as shown in Fig. 3(b), increases linearly with time until $t = T$ after which it remains constant at $(a + b_0T)$. From the stability proof, this implies that the trajectories of s will converge in a finite time to a ball of radius $\gamma(T)$ centered at the origin. The radius $\gamma(T)$ is defined from (17) as

$$\gamma(T) \triangleq \frac{1}{2(a + b_0T)} \ln \left(\frac{1 + \alpha}{1 - \alpha} \right) \quad (18)$$

and provides a bound on the scalar variable s . This translates to a bound on the errors e and \dot{e} [8] in terms of the parameters a, b_0, T , and λ . Therefore (18) provides the scope for the proper selection of these parameters.

In this practical implementation, the boundary layer thickness $\gamma(t)$ decreases with time over the interval $0 \leq t \leq T$, after which it remains constant at the value $\gamma(T)$ defined by (18). At the initial point of time the boundary layer thickness will be large and this will be useful in eliminating chattering in systems where the uncertainty¹ is initially large, as for example in systems with an adaptation law for the cancellation of uncertainties. After time T , the smooth compensator discussed here will be similar to the saturation compensator in the sense that both have boundary layers of fixed thickness.

IV. COMPARATIVE NUMERICAL SIMULATIONS

In order to reveal the superiority of the smooth time-varying compensator over the conventional bang-bang and saturation compensators, we consider the tracking control of the SDOF uncertain mechanical system discussed in Sections II and III. The desired trajectory of the sdof system is chosen to be $x_d = -0.3 + 0.3\cos(t)$ such that the desired position and desired velocity at time $t = 0$ are zero. The true value and the estimated value of the generalized inertia are $M = 2.0$ and $\hat{M} = 1.5$, respectively. The upper bound on the uncertainty is taken to be $\beta = 1.0$.

The simulation duration was set to 10 s. The controllers used for the comparison of the bang-bang compensator, the saturation compensator and the smooth time-varying compensator are given by (3), (7), and (9), respectively. The parameter b in (9) was chosen to be constant. It can be seen from these equations that the controllers have the same structure and differ only in the robust compensator term. The bang-bang compensator has no boundary layer. In order to have a fair comparison between the saturation compensator and the smooth compensator, we choose $a = 100$ and $b = 10$ such that the smooth compensator approximates the saturation compensator with a boundary layer thickness $\epsilon = 0.005$ at time $t = 0$. The choice of a and b is made in accordance with (17).

The errors in the position, velocity, and acceleration for the three cases are shown in Figs. 4–6, respectively. The control action generated by the different compensators are

¹In the example we considered here, ρ represents the uncertainty. It is defined in (4) and is used in the control law in (9)

shown in Fig. 7. From Fig. 4 it becomes obvious that the bang-bang compensator results in chattering though it achieves the best accuracy. The smooth compensator and the continuous compensator have almost the same error initially but as time progresses, the error diminishes with time in the case of the smooth compensator. The superiority of the smooth compensator becomes clear from Figs. 5 and 6 where the errors in the velocity and the acceleration are found to be the most smooth in the case of the smooth compensator. Also, the smooth compensator results in the least acceleration error. The control actions in Fig. 7 indicate that the smooth compensator generates smooth control action while the conventional compensators result in violent chattering.

V. CONCLUSION

In this paper we carry out a comparative study of conventional bang-bang and saturation compensators with a smooth time-varying robust compensator employing a hyperbolic tangent function. Both theoretical analysis and numerical simulations indicate the superiority of the smooth robust compensator over the conventional compensators. The smooth compensator employs a boundary layer to reduce chattering and the time-varying nature of the boundary layer guarantees the asymptotic stability of the closed-loop system.

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